Local discontinuous Galerkin methods for high order derivative equations

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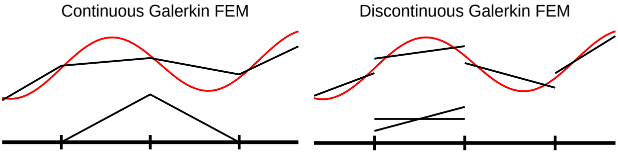
Outline

Discontinuous Galerkin Methods

- Finite element method for approximating PDE.
- Piecewise polynomial completely discontinuous.



- Local variational formulation (element-by-element).
- First introduced in 1973 by Reed and Hill.
- Hyperbolic conservation law by Cockburn and Shu.
- According the search in Mathscinet, papers with key words "Discontinuous Galerkin"
  - Before 2000, 203 papers;
  - 2001-2014, 2357 papers.

Outline

1. Introduction to discontinuous Galerkin (DG) methods
   - DG method for hyperbolic conservation laws
   - LDG method for heat equation
2. LDG method for high order derivative equations
3. Theoretical analysis of LDG methods
4. Time discretization
5. Fast solver for implicit system
   - Cahn-Hilliard equation
   - KdV type equations
6. Conclusion and future work

Introduction to DG method for hyperbolic conservation laws

To solve a conservation law:

$$u_t + f(u)_x = 0.$$

Assume the solution u come from a finite dimensional approximation space V_h, which is usually taken as the space of piecewise polynomials of the degree up to k:

$$V_h = \{v : v|_{I_j} \in P^k(I_j), j = 1, \cdots, N\},$$

where $I_j = [x_{j-1/2}, x_{j+1/2}]$. Notice that $u \in V_h$ is discontinuous (double-valued) at the cell interfaces.

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Multiplying with a test function $v \in V_h$ and integrating by parts over a cell $I_j = [x_{j-1/2}, x_{j+1/2}]$, we have

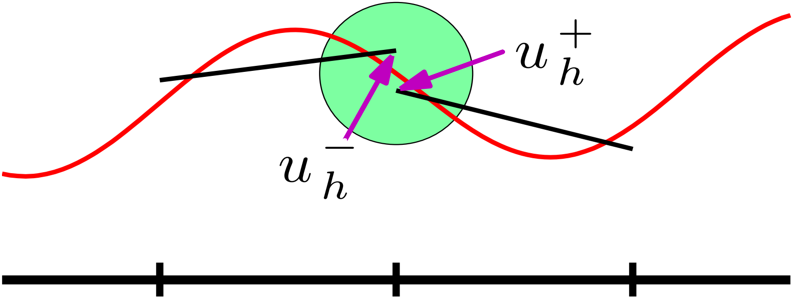
$$\int_{I_j} u_t v dx - \int_{I_j} f(u) v_x dx + f(u_{j+\frac{1}{2}}) v_{j+\frac{1}{2}} - f(u_{j-\frac{1}{2}}) v_{j-\frac{1}{2}} = 0.$$

However, the boundary terms $f(u_{j+\frac{1}{2}})$ and $v_{j+\frac{1}{2}}$ etc. are not well-defined when $u, v \in V_h$, as they are discontinuous (double-valued) at the cell interfaces.

Double−valued, need to pick/define one

$$\widehat{f(u_h)} = \widehat{f}(u_h^-, u_h^+)$$

$$\widehat{u}_h = \widehat{u}(u_h^-, u_h^+)$$



From the conservation and stability (upwind) considerations, we take

- A single value monotone numerical flux to replace $f(u_{j+\frac{1}{2}})$:

$$\hat{f}_{j+\frac{1}{2}} = \hat{f}(u^-_{j+\frac{1}{2}}, u^+_{j+\frac{1}{2}})$$

  where $\hat{f}(u,u) = f(u)$(consistency); $f(\uparrow,\downarrow)$ (monotonicity) and $\hat{f}$ is Lipschitz continuous with respect to both arguments.

- Values from inside I_j for the test function v:

$$v^+_{j-\frac{1}{2}}, \quad v^-_{j+\frac{1}{2}}$$

DG scheme

Hence the DG scheme is: Find $u \in V_h$ such that

$$\int_{I_j} u_t v dx - \int_{I_j} f(u) v_x dx + \hat{f}_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}} = 0.$$

for all $v \in V_h$.

Time discretization: TVD Runge-Kutta method(Shu and Osher, JCP 88)

For the semi-discrete scheme:

$$\frac{du}{dt} = L(u)$$

where $L(u)$ is a spatial discretization operator, the third order TVD Runge-Kutta method is simply:

$$\begin{aligned}
u^{(1)} &= u^n + \Delta t L(u^n) \\
u^{(2)} &= \frac{3}{4}u^n + \frac{1}{4}(u^{(1)} + L(u^{(1)})) \\
u^{n+1} &= \frac{1}{3}u^n + \frac{2}{3}(u^{(2)} + L(u^{(2)}))
\end{aligned}$$

Advantages of DG methods:

- ✓ Wide Range of PDE's
- ✓ Easy handling complicated geometry and boundary conditions
- ✓ Allowing the hanging nodes
- ✓ Compact and then parallel efficiency.
- ✓ Easy $h-p$ adaptivity;
- ✓ Flexible choice of approximation spaces

Disadvantages of DG methods:

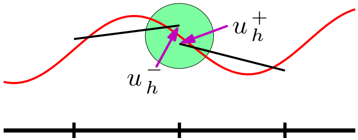
- × more of degrees of freedom
- × Systems of equations difficult to solve
- × Techniques under development

Numerical fluxes

Double−valued, need to pick/define one

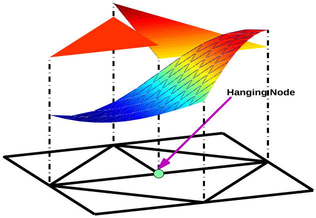
$$\widehat{f(u_h)} = \widehat{f}\,(u_h^-, u_h^+)$$

$$\widehat{u}_h = \widehat{u}\,(u_h^-, u_h^+)$$



Hanging node

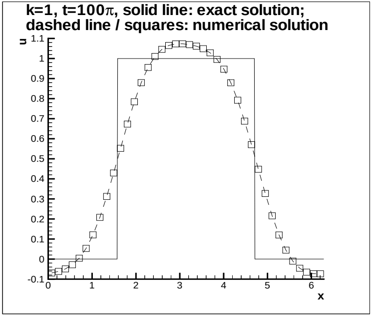
Nonconforming Mesh and Variable Degree

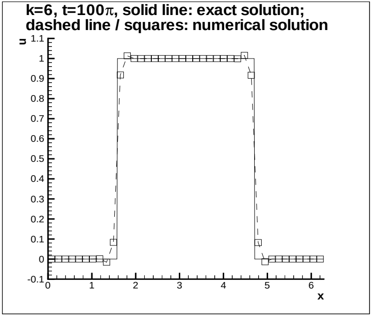


Example: linear convection equation in 1D

$$u_t + u_x = 0$$

on the domain $(0;2\pi) \times (0;T)$with the characteristic function of the interval $(\frac{\pi}{2};\frac{3\pi}{2})$ as initial condition and periodic boundary conditions with 40 cells (Cockburn-Shu, JSC, 2001).

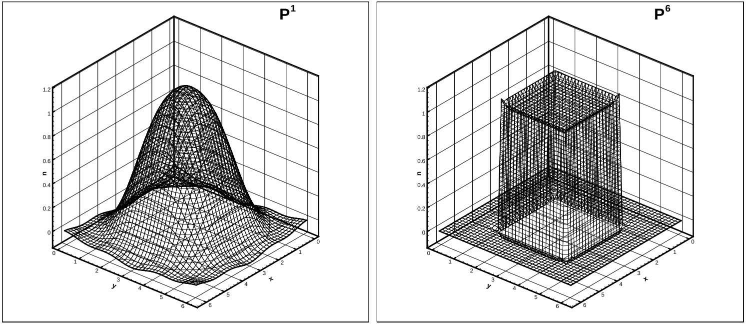




Example: linear convection equation in 2D

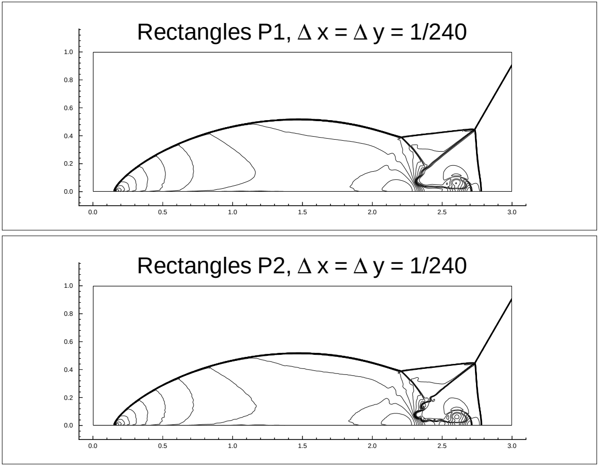
$$u_t + u_x + u_y = 0$$

on the domain $(0; 2\pi)^2 \times (0; T)$with the characteristic function of the interval $(\frac{\pi}{2}; \frac{3\pi}{2})^2$ as initial condition and periodic boundary conditions with 40×40 cells (Cockburn-Shu, JSC, 2001).



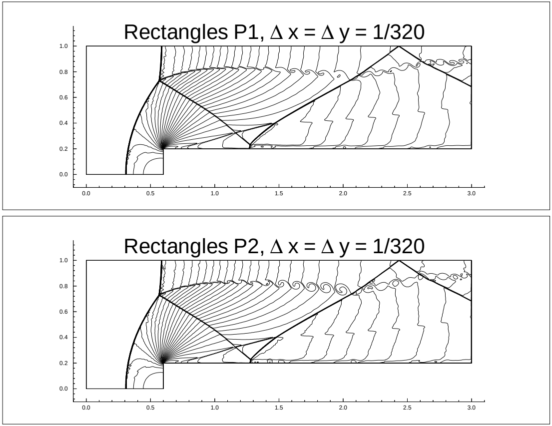
Two dimensional compressible Euler equations

Double Mach reflection problem (Cockburn-Shu, JSC, 2001)



Two dimensional compressible Euler equations

The flow past a forward-facing step problem. No special treatment is performed near the corner singularity. (Cockburn-Shu, JSC, 2001)



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Difficulty in generalizing DG to PDEs containing higher spatial derivatives

Generalization the DG method to PDEs containing higher spatial derivatives. For example, the heat equation

$$u_t - u_{xx} = 0$$

with proper boundary and initial conditions.

A straightforward generalization is replacing $f(u) = -u_x$ in the DG scheme for the conservation law ($u_t + f(u)_x = 0$): find $u \in V_h$ such that, for all test functions $v \in V_h$,

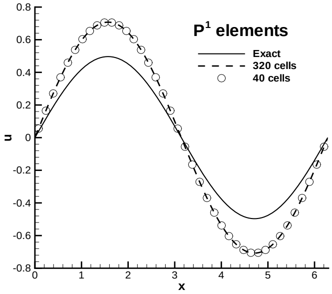
$$\int_{I_j} u_t v dx + \int_{I_j} u_x v_x dx - \widehat{u_x}_{j+\frac{1}{2}} v_{j+\frac{1}{2}} + \widehat{u_x}_{j-\frac{1}{2}} v_{j-\frac{1}{2}} = 0.$$

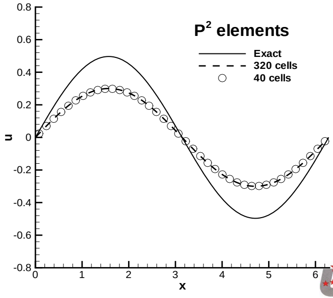
Considering that diffusion is isotropic, a nature choice of the flux could be the central flux

$$\widehat{u_x}_{j+\frac{1}{2}} = \frac{1}{2}\left((u_x)^-_{j+\frac{1}{2}} + (u_x)^+_{j+\frac{1}{2}}\right)$$

However, it has been proven in Zhang and Shu, M^3AS 03 that the scheme is

- Consistent with the heat equation
- (very weakly) unstable





The LDG method for the heat equation (Bassi and Rebay, JCP 97; Cockburn and Shu, SINUM 98):

- Rewrite the heat equation as

$$u_t - q_x = 0, \quad q - u_x = 0.$$

- Find $u, q \in V_h$ such that, for all $v, w \in V_h$,

$$\int_{I_j} u_t v dx + \int_{I_j} q v_x - \hat{q}_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} + \hat{q}_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}} = 0,$$
$$\int_{I_j} q p dx + \int_{I_j} u p_x - \hat{u}_{j+\frac{1}{2}} p^-_{j+\frac{1}{2}} + \hat{u}_{j-\frac{1}{2}} p^+_{j-\frac{1}{2}} = 0.$$

q can be locally solved and eliminated, hence local DG.

The choice for the numerical flux is the following alternated flux

$$\hat{u}_{j+\frac{1}{2}} = u^{-}_{j+\frac{1}{2}}, \quad \hat{q}_{j+\frac{1}{2}} = q^{+}_{j+\frac{1}{2}},$$

or

$$\hat{u}_{j+\frac{1}{2}} = u^{+}_{j+\frac{1}{2}}, \quad \hat{q}_{j+\frac{1}{2}} = q^{-}_{j+\frac{1}{2}}.$$

Then we have

- L^2 stability
- optimal convergence of $\mathcal{O}(h^{k+1})$ in L^2 for P^k elements for u and q.

Table: L^2 and L^∞ errors and orders of accuracy for the LDG method with alternated fluxes applied to the heat equation with an initial condition $u(x,0) = \sin(x)$, $t = 1$. Third order Runge-Kutta in time with a small Δt so that time error can be ignored.

	$k=1$				$k=2$			
Δx	L^2 error	order	L^∞ error	order	L^2 error	order	L^∞ error	order
$2\pi/20$, u	1.58E-03	—	6.01E-03	—	3.98E-05	—	1.89E-04	—
$2\pi/20$, q	1.58E-03	—	6.01E-03	—	3.98E-05	—	1.88E-04	—
$2\pi/40$, u	3.93E-04	2.00	1.51E-03	1.99	4.98E-06	3.00	2.37E-05	2.99
$2\pi/40$, q	3.94E-04	2.00	1.51E-03	1.99	4.98E-06	3.00	2.37E-05	2.99
$2\pi/80$, u	9.83E-05	2.00	3.78E-04	2.00	6.22E-07	3.00	2.97E-06	3.00
$2\pi/80$, q	9.83E-05	2.00	3.78E-04	2.00	6.22E-07	3.00	2.97E-06	3.00
$2\pi/160$, u	2.46E-05	2.00	9.45E-05	2.00	7.78E-08	3.00	3.71E-07	3.00
$2\pi/160$, q	2.46E-05	2.00	9.45E-05	2.00	7.78E-08	3.00	3.71E-07	3.00

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Main idea of LDG method for high order derivative equations

- Rewrite the high order derivative term into the proper first order equations.
- Use the DG method for the first order equations.
- The key point of the method is to design the numerical fluxes to ensure the stability.
  - Odd derivatives equation: upwinding principle.
  - Even derivatives equation: alternating fluxes.

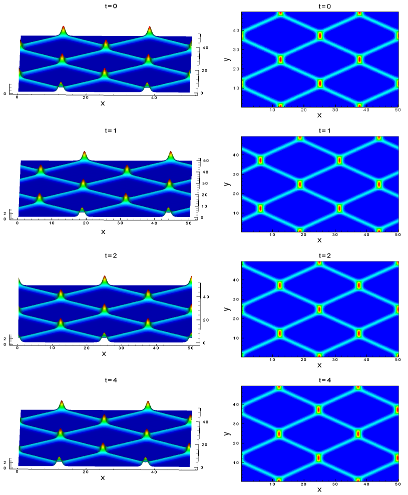
Review paper

- Y. Xu and C.-W. Shu, Local discontinuous Galerkin methods for high-order time-dependent partial differential equations, Communications in Computational Physics, 7 (2010), pp. 1-46.

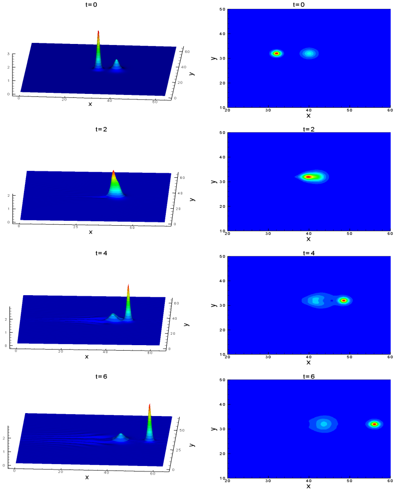
LDG methods for nonlinear dispersive equations

- KdV equation (Yan and Shu SINUM 2002, Xu-Shu CMAME 2007).
- KdV-Burgers equation, Kawahara equation (Xu-Shu, JCM 2004).
- Fully nonlinear $K(m, n)$ and $K(n, n, n)$ equations(Levy-Shu-Yan JCP 2004, Xu-Shu JCM 2004).
- Kadomtsev-Petviashvili equation (Xu-Shu, Physica D 2005).
- Zakharov-Kuznetsov equation (Xu-Shu Physica D 2005, Xu-Shu CMAME 2007).
- Ito-type coupled KdV equations (Xu-Shu CMAME 2006).

Kadomtsev-Petviashvili equation (Physica D, 2005)



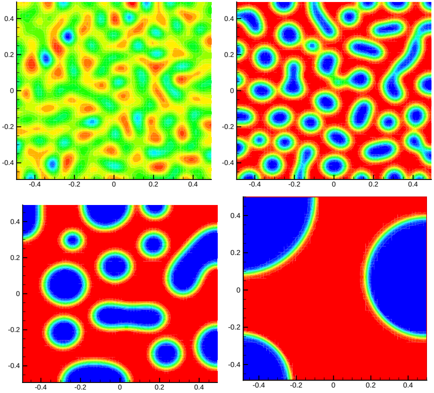
Zakharov-Kuznetsov equation (Physica D, 2005)



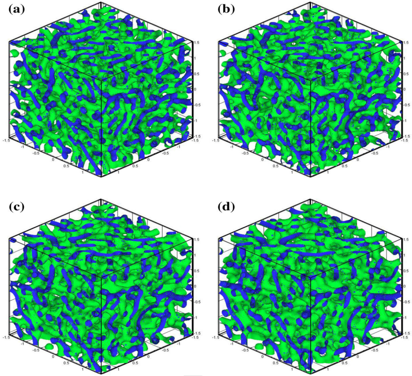
LDG methods for phase field models

- Cahn-Hilliard equation (Xia-Xu-Shu JCP 2007, Guo-Xu JSC 2014)
- Allen-Cahn/Cahn-Hilliard system (Xia-Xu-Shu, CICP 2009)
- Functionalized Cahn-Hilliard equation (Guo-Xu-Xu, JSC 2015)
- No-slop-selection thin film model (Xia, JCP 2015)
- Cahn-Hilliard-Hele-Shaw system (Guo-Xia-Xu, JCP 2014)
- Cahn-Hilliard-Brinkman system (Guo-Xu, JCP 2015)
- Phase field crystal equation (Guo-Xu, submitted)

2D Cahn-Hilliard equation (JSC, 2014)



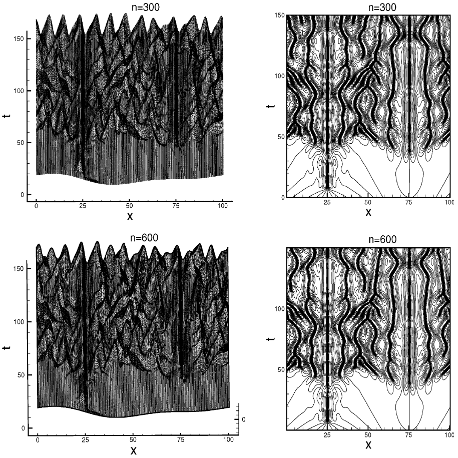
3D Functionalized Cahn-Hilliard (JSC, 2015)



LDG methods for nonlinear diffusion equations

- Bi-harmonic equations (Yan-Shu JSC 2002, Dong-Shu SINUM 2009).
- Kuramoto-Sivashinsky equation (Xu-Shu, CMAME 2006).
- Surface diffusion of graphs and Willmore flow of graphs (Xu-Shu JSC 2009, Ji-Xu submitted 2009).
- Porous medium equation (Zhang-Wu JSC 2009).

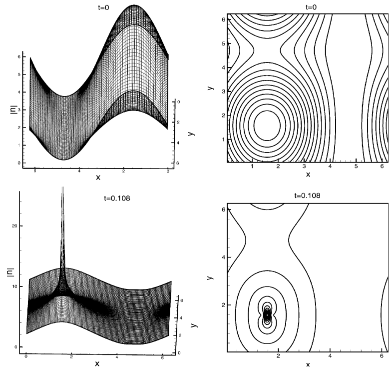
Kuramoto-Sivashinsky (CMAME 2006)



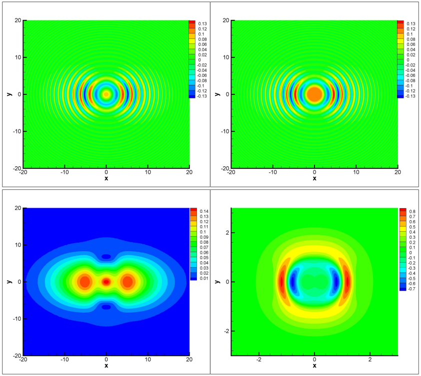
LDG methods for Schrödinger equation

- Nonlinear Schrödinger equations (Xu-Shu JCP 2005, Lu-Cai-Zhang IJAM 2005)
- Zakharov system (Xia-Xu-Shu JCP 2010)
- Stationary Schrödinger equations (Wang-Shu JSC 2009, Guo-Xu CICP 2014)
- Nonlinear Schrödinger-KdV System (Xia-Xu-Shu CICP 2014)
- Nonlinear Schrödinger equation with wave operator (Guo-Xu JSC 2015)

2D Schrödinger equation (JCP, 2005)

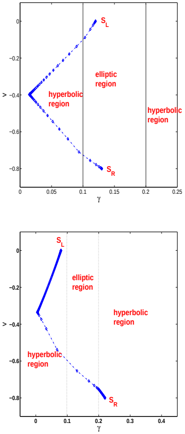


2D Zakharov system (JCP, 2010)

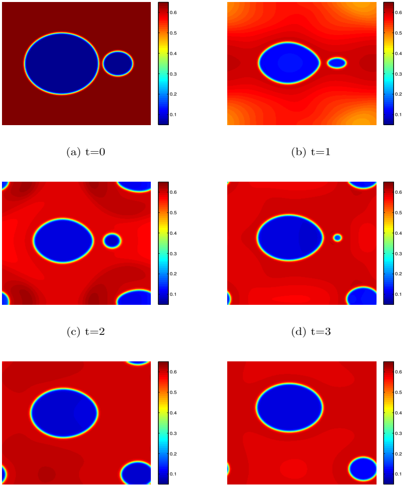


LDG methods for phase transition problems

1D phase transition in solid (JSC 2014)



Navier-Stokes-Korteweg (JCP, 2015)



(a) t=0 (b) t=1

(c) t=2 (d) t=3

LDG methods for other equations

- Degasperis-Procesi (DP) equation (Xu-Shu, CICP 2011).

$$u_t - u_{xxt} + 4uu_x = 3u_x u_{xx} + uu_{xxx}$$

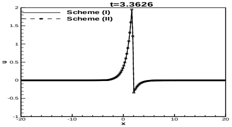
- Camassa-Holm (CH) equation (Xu-Shu, SINUM 2008).

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}.$$

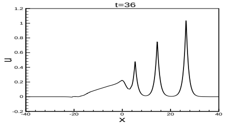
- Hunter-Saxton (HS) equation (Xu-Shu, SIJSC 2008 and JCM 2010).

$$u_{xxt} + 2u_x u_{xx} + uu_{xxx} = 0$$

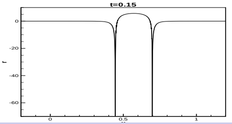
Degasperis-Procesi (CICP)



Camassa-Holm (SINUM)



Hunter-Saxton (SIJSC)



Theoretical analysis of LDG methods

- Energy stability
- L^2 *a priori* error estimates
- Negative order norm estimates.

Semi-discrete energy stability

- General nonlinear case ✓
- Exception: Navier-Stokes-Korteweg equation ?

Fully-discrete energy stability

- Crank-Nicholson scheme on time
  - Nonlinear Schrödinger dinger equation with wave operator (Guo-Xu JSC 2015)
- Implicit backward Euler scheme on time with convex splitting
  - Cahn-Hilliard equations (Guo-Xu JSC 2014)
  - Cahn-Hilliard-Hele-Shaw system (Guo-Xia-Xu JCP 2014)
  - Cahn-Hilliard-Brinkman (Guo-Xu JCP 2015)
  - Phase field crystal equation (Guo-Xu submitted 2015)
- IMEX scheme on time
  - Advection-diffusion problem (Wang-Shu-Zhang SINUM 2015)

Semi-discrete error estimates

Assume u is the exact smooth solution, $u_h \in V_h$ is the solution computed by LDG methods.

- L^2 *a priori* error estimates

$$\|u - u_h\|_{L^2} \le Ch^{k+1}?$$

- Negative order norm estimates.

$$\|u - u_h\|_{-\ell,\Omega_0} \le C\, h^{2k+1}?$$

where the negative order norm is defined as $\|u\|_{-\ell,\Omega} = \sup_{\Phi \in C_0^\infty(\Omega)} \frac{(u,\Phi)_\Omega}{\|\Phi\|_{\ell,\Omega}}$.

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Main difficulty of L^2 *a priori* error estimates

- Nonlinear term.
- Lack of control on some of the jump terms at cell boundaries for high order derivatives term.
- Special projection is introduced to handle troublesome jump terms in the error equation.
- It is more challenging to perform L^2 *a priori* error estimates for PDEs with high order derivatives than for first order hyperbolic PDEs.

Sub-optimal L^2 *a priori* error estimates $\mathcal{O}(h^{k+\frac{1}{2}})$

- Nonlinear KdV equations (1D): (Xu-Shu CMAME 2007).
- Nonlinear Zakharov-Kuznetsov equation (2D): (Xu-Shu CMAME 2007).
- Camassa-Holm equations (1D): (Xu-Shu SINUM 2008).

Optimal L^2 *a priori* error estimates $\mathcal{O}(h^{k+1})$

- Bi-harmonic equations (Multi-D): (Dong-Shu SINUM 2009)
- Nonlinear Willmore flow equations (Multi-D): (Ji-Xu IJNAM 2011)
- Linear odd order equations (1D): (Xu-Shu SINUM 2012)
- Linear Schrödinger equations (Multi-D): (Xu-Shu SINUM 2012)
- Nonlinear surface diffusion equations (Multi-D): (Ji-Xu JSC 2012)
- Nonlinear Schrödinger equation with wave operator (Multi-D): (Guo-Xu JSC 2015)

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Negative order norm estimates

$$\|u - u_h\|_{-\ell,\Omega_0} \leq C\, h^{2k+1}$$

where $\|u\|_{-\ell,\Omega} = \sup_{\Phi \in C_0^\infty(\Omega)} \frac{(u,\Phi)_\Omega}{\|\Phi\|_{\ell,\Omega}}$.

- Linear convection diffusion equation: (Ji-Xu-Ryan MC 2012)
- Nonlinear hyperbolic conservation laws: (Ji-Xu-Ryan JSC 2013)
- Linear odd order equations ?

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Explicit time discretization

- Easy implementation
- The time step $\Delta t = c\Delta x^k$, k is the order of derivatives.

TVD Runge-Kutta method(Shu and Osher, JCP 88)

For the semi-discrete scheme:

$$\frac{du}{dt} = L(u)$$

where $L(u)$ is a spatial discretization operator, the third order TVD Runge-Kutta method is simply:

$$\begin{aligned} u^{(1)} &= u^n + \Delta t L(u^n) \\ u^{(2)} &= \frac{3}{4}u^n + \frac{1}{4}(u^{(1)} + L(u^{(1)})) \\ u^{n+1} &= \frac{1}{3}u^n + \frac{2}{3}(u^{(2)} + L(u^{(2)})) \end{aligned}$$

Implicit time dsicreization

- Additive Runge-Kutta (ARK) method
- Spectral Deferred Correction (SDC) method
- Diagonally Implicit Runge-Kutta (DIRK) method
- Implicit-explicit (IMEX) method
- Exponential Time Differencing (ETD) methods

- Solve linear/nonlinear system $A(u) = f$.
- Large time step: $\Delta t = c\Delta x$.

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5. Fast solver for implicit system
   - Cahn-Hilliard equation
   - KdV type equations
6. Conclusion and future work

Motivation

- Higher order discontinuous Galerkin methods provide accurate discretizations of time-dependent partial differential equations with higher order spatial derivatives.
- Explicit stability constraint of the DG method applied to the higher order spatial derivatives decreases dramatically.
- The time step $\Delta t = c\Delta x^k$, k is the order of derivatives.

Objectives

- Higher order DG space discretization
- High order implicit time discretization
- Fast solver to solve the discretization system $Au = f$

Linear Matrix A

- large
- sparse
- ill-conditioned

Solver

- Direct Methods (LAPACK): Cost$=O(N^3)$, $N \longrightarrow \infty$
- Iterative Methods
  - Multigrid (MG) method
  - Nonlinear Full Approximation Scheme (FAS) multigrid methods

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The Multigrid algorithm

Two-Grid Correction Scheme

- An initial function $u_h^0 = u_{h,PRE}^0$ on the finer grid
- Apply ν_1 pre-relaxation sweeps:

$$u_{h,PRE}^{i+1} = u_{h,PRE}^{i} + B_h(f_h - A_h u_{h,PRE}^{i})$$

- Update the solution by a coarse-grid correction step

$$u_{h,POST}^{0} = u_{h,PRE}^{\nu_1} + P_{hH} A_H^{-1} R_{Hh}(f_h - A_h u_{h,PRE}^{\nu_1})$$

- Apply ν_2 post-relaxation sweeps

$$u_{h,POST}^{i+1} = u_{h,POST}^{i} + B_h(f_h - A_h u_{h,POST}^{i})$$

where B_h is an approximate inverse of A_h. P_{hH} and R_{Hh} are prolongation and restriction operator respectively.

Possible choice of smoother operator

We decompose A_h into a strict block-lower, block-diagonal, and strict block-upper matrix, i.e.

$$A_h = L_h + D_h + U_h.$$

- Block-Jacobi smoother: $B_h = D_h^{-1}$.
- Block Gauss-Seidel smoother: $B_h = (D_h + L_h)^{-1}$.
- Damped block-Jacobi smoother: $B_h = \alpha D_h^{-1}$.
- Damped block Gauss-Seidel smoother: $B_h = \alpha(D_h + L_h)^{-1}$.

Local model analysis

- The convergence factor of the two-grid method (A. Brandt, SINUM, 1994).

$$\lambda = \sup \frac{\|u_{POST}^{\nu_2}\|}{\|u_{PRE}^{0}\|}$$

- The convergence factor might be computed in terms of the symbol of the error propagation operator.
- For the linear iteration, the error propagation operator is defined as:

$$E_h = I - B_h A_h.$$

- When $\lambda < 1$, we will ge a convergent iteration.
- The smaller λ is, the faster is the iteration.

Algorithm 1

- The error propagation reads:

$$E_h^{2grid} := E_h^{\nu_2}[I - P_{hH}A_H^{-1}R_{Hh}A_h]E_h^{\nu_1}.$$

- $\lambda = \sup_{\theta \neq 0} \|\widehat{E}_h(\theta)\|$, where $\widehat{E}_h(\theta)$ denote the error propagation operator in the frequency space.
- For symmetric problems, the estimation of the spectral radius of E_h could be reduced to the computation of its largest eigenvalue.
- For non-symmetric case, one can define the *asymptotic convergence factor* as:

$$\lambda_{asymp} = \sup_{\theta \neq 0} \sigma_1(\widehat{E}_h(\theta)).$$

where σ_1 is the spectral radius of E, (i.e. largest absolute eigenvalue).

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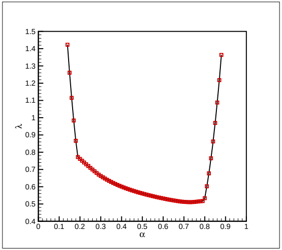
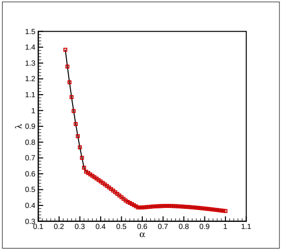
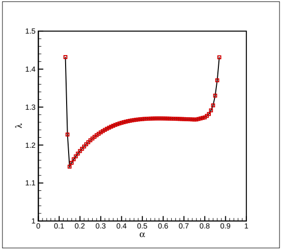
Asymptotic convergence factor changes with damping parameter α in 1D



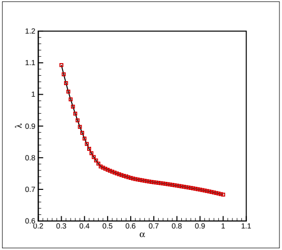
(a) N=64, Jacobi, $\mathcal{P}^1$



(b) N=64, Gauss-Seidel, $\mathcal{P}^1$

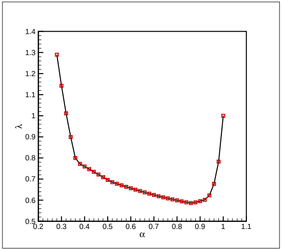


(c) N=64, Jacobi, $\mathcal{P}^2$

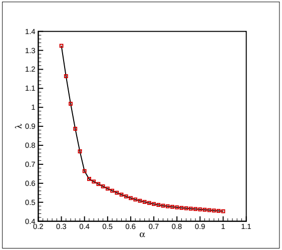


(d) N=64, Gauss-Seidel, $\mathcal{P}^2$

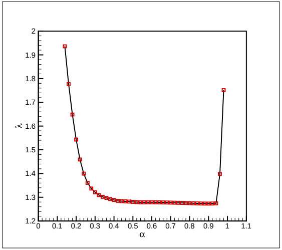
Asymptotic convergence factor changes with damping parameter α in 2D



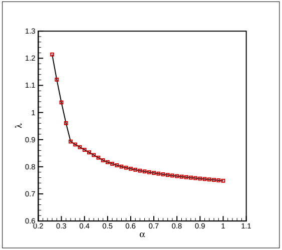
(a) N=64, Jacobi, $\mathcal{P}^1$



(b) N=64, Gauss-Seidel, $\mathcal{P}^1$

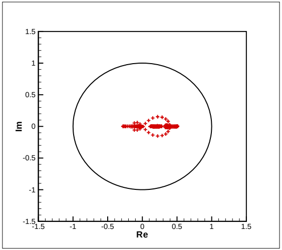


(c) N=64, Jacobi, $\mathcal{P}^2$

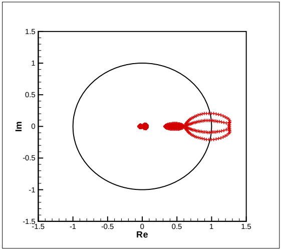


(d) N=64, Gauss-Seidel, $\mathcal{P}^2$

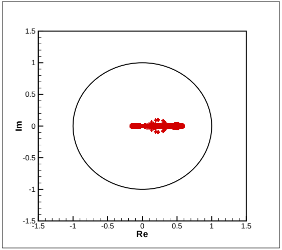
Eigenvalue spectral of E_h^{2grid} with damped Jacobi smoother



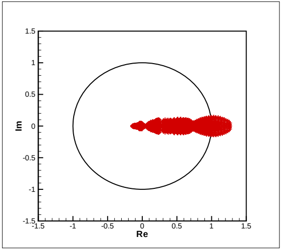
(a) E_h^{2grid}, $\alpha = 0.75$, $\mathcal{P}^1$, 1D



(b) E_h^{2grid}, $\alpha = 0.75$, $\mathcal{P}^2$, 1D

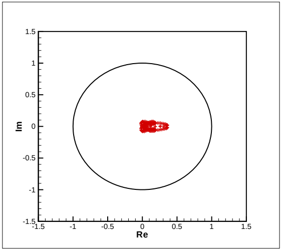


(c) E_h^{2grid}, $\alpha = 0.85$, $\mathcal{P}^1$

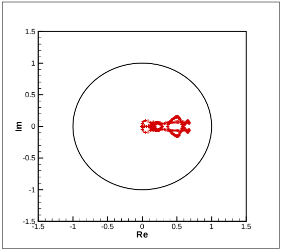


(d) E_h^{2grid}, $\alpha = 0.85$, $\mathcal{P}^2$, 2D

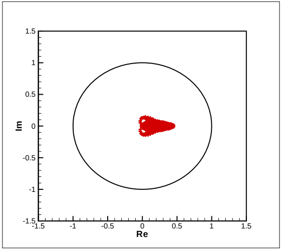
Eigenvalue spectral of E_h^{2grid} with damped Gauss-Seidel



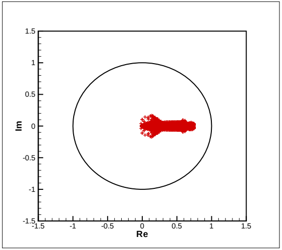
(a) E_h^{2grid}, $\alpha = 1.0$, $\mathcal{P}^1$, 1D



(b) E_h^{2grid}, $\alpha = 1.0$, $\mathcal{P}^2$, 1D

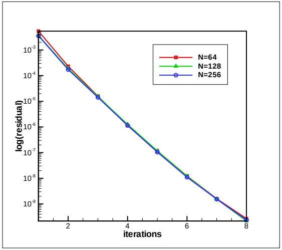


(c) E_h^{2grid}, $\alpha = 1.0$, $\mathcal{P}^1$, 2D

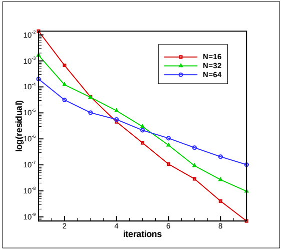


(d) E_h^{2grid}, $\alpha = 1.0$, $\mathcal{P}^2$, 2D

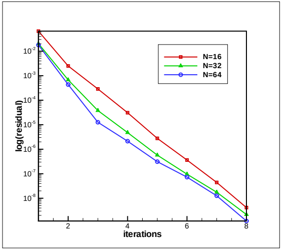
Convergence rate of FAS Multigrid solver



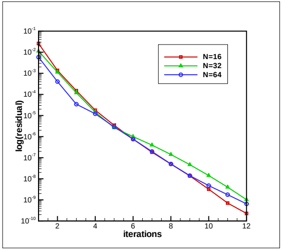
(a) Gauss-Seidel, P^1, 2D



(b) Gauss-Seidel, P^2 , 2D



(c) Gauss-Seidel, P^1, 3D



(d) Gauss-Seidel, P^2 , 3D

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Motivation

- The MG method is mainly used to solve PDEs with even-order spatial derivatives.
- There is little work for PDEs with odd-order spatial derivatives.
- Does the MG method work for this type of equations?
- Consider the one-dimensional linear KdV equation

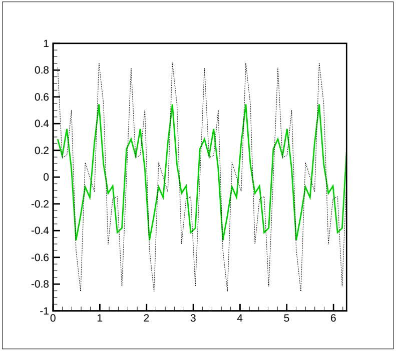
$$u_t + u_{xxx} = 0.$$

- We use an initial guess

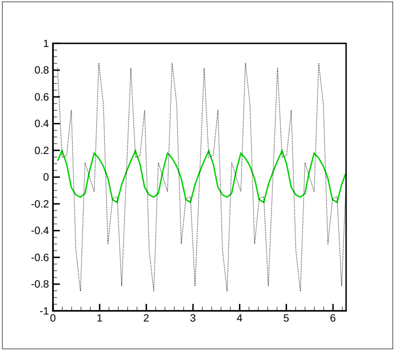
$$u_j^h = \frac{1}{2}[\sin(\frac{16j\pi}{n}) + \sin(\frac{40j\pi}{n})],$$

consisting of the $k = 16$ and $k = 40$ modes.

Coarse-grid correction on a grid with $n = 64$



(a) The error after one sweep of weighted Jacobi.



(b) The error after three sweeps of weighted Jacobi.

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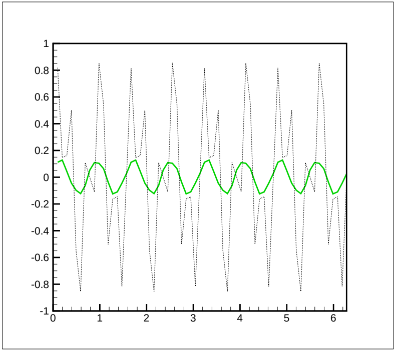
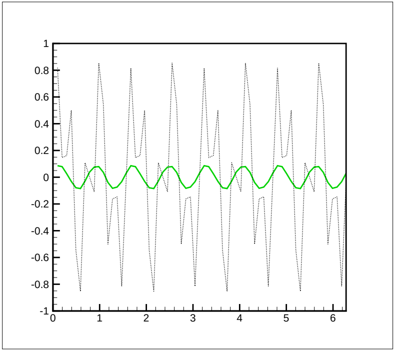
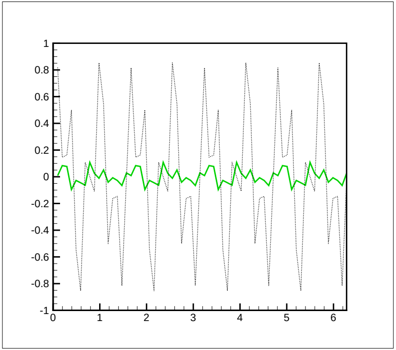


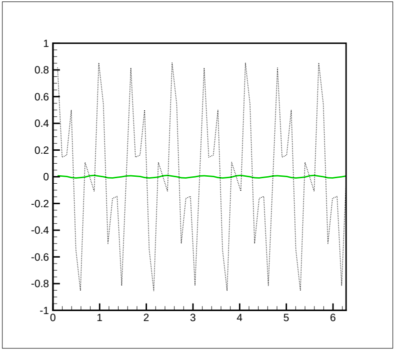
(c) The error after four sweeps of weighted Jacobi.



(d) The error after five sweeps of weighted Jacobi.



(e) The fine-grid error after three sweeps of weighted Jacobi on the coarse-grid problems.



(f) The fine-grid error after the coarse-grid correction is followed by three weighted Jacobi sweeps on the fine grid.

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The LDG scheme for the KdV type equations

Rewrite equation as a first order system:

$$u_t + p_x = 0, \quad p - q_x = 0, \quad q - u_x = 0.$$

Applying the LDG method to the system, we obtain an ODE system and apply the time marching method.

$$\mathbf{u}_t + M_R\mathbf{p} = 0, \quad \mathbf{p} - M_R\mathbf{q} = 0, \quad \mathbf{q} - M_L\mathbf{u} = 0. \tag{1}$$

Time discretization and solver

- High order implicit ARK time discretization method.
- Multigrid method to solve linear system.
- Time step $\Delta t = \mathcal{O}(\Delta x)$.

Two methods to eliminate the auxiliary variables (I)

- **Method 3X**

  Eliminate q and p and get an ODE

$$\mathbf{u}_t = L(\mathbf{u}).$$

The backward Euler time marching method is applied and we obtain a linear system

$$A\mathbf{u}^{n+1} = \mathbf{f},$$

where $A = I + \Delta t M_R^2 M_L$, f is the corresponding right hand side vector consisting of u^n and I is the identical matrix.

Two methods to eliminate the auxiliary variables (II)

- **Method X-2X**

  Only eliminate q, then we get

$$\begin{cases} \mathbf{u}_t = L_1(\mathbf{p}), \\ \mathbf{p} = L_2(\mathbf{u}). \end{cases} \tag{2}$$

We apply the backward Euler time marching method and obtain a system of two coupled equations for $[\mathbf{u}^{n+1}, \mathbf{p}^{n+1}]$

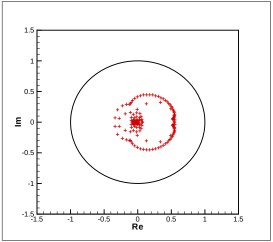
$$G\mathbf{U} = \mathbf{F},$$

where

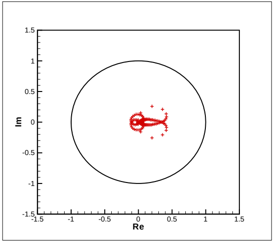
$$\mathbf{U} = \begin{bmatrix} \mathbf{u}^{n+1} \\ \mathbf{p}^{n+1} \end{bmatrix}, \quad G = \begin{bmatrix} I & \Delta t M_R \\ -M_R M_L & I \end{bmatrix}$$

and F is the corresponding right hand side vector consisting of $\mathbf{u}^n$ and $\mathbf{p}^n$.

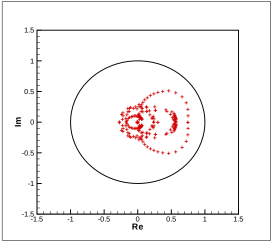
Convergence behavior for the multigrid method



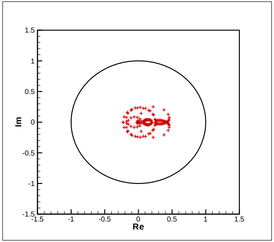
(a) 3X, $\mathcal{P}^1$ E_h^{2grid}, $\alpha = 0.7$



(b) X-2X, $\mathcal{P}^1$ E_h^{2grid}, $\alpha = 0.7$



(c) 3X, $\mathcal{P}^2$ E_h^{2grid}, $\alpha = 0.8$

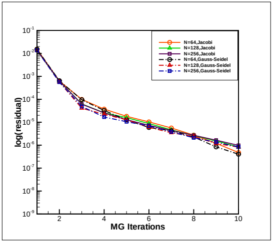


(d) X-2X, $\mathcal{P}^2$ E_h^{2grid}, $\alpha =$

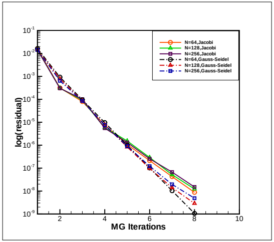
Accuracy test for the equation $u_t + u_{xxx} = 0$, $\Delta t = 0.1\Delta x$.

		Method 3X				Method X-2X			
	N	L^2 error	order	L^∞ error	order	L^2 error	order	L^∞ error	order
$\mathcal{P}^1$	16	8.54E-02	–	6.59E-02	–	8.54E-02	–	6.60E-02	–
	32	1.83E-02	2.22	2.08E-02	1.66	1.83E-02	2.22	2.08E-02	1.66
	64	4.36E-03	2.07	5.77E-03	1.85	4.36E-03	2.07	5.77E-03	1.85
	128	1.07E-03	2.01	1.50E-03	1.94	1.07E-03	2.01	1.50E-03	1.94
$\mathcal{P}^2$	16	6.99E-03	–	5.94E-03	–	6.99E-03	–	5.94E-03	–
	32	8.78E-04	2.99	7.90E-04	2.91	8.78E-04	2.99	7.90E-04	2.91
	64	1.10E-04	2.99	1.00E-04	2.98	1.10E-04	2.99	1.00E-04	2.98
	128	1.38E-05	3.00	1.26E-05	2.99	1.38E-05	3.00	1.26E-05	2.99

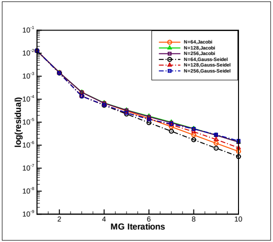
Convergence rates for the MG solver for $\mathcal{P}^1$ and $\mathcal{P}^2$



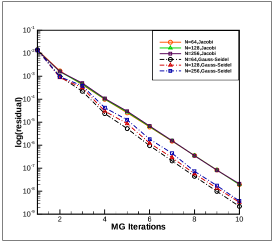
(a) Method 3X, $\mathcal{P}^1$



(b) Method X-2X, $\mathcal{P}^1$



(c) Method 3X, $\mathcal{P}^2$



(d) Method X-2X, $\mathcal{P}^2$

Observation

- High order time discretization methods can be coupled with the LDG space discretization.
- The eigenvalue spectra of E_h^{2grid} is strictly less than 1, i. e. the two-grid algorithm is convergent.
- Method X-2X shows better convergence behavior than Method 3X.
- Method X-2X require more memory, especially for high dimensional case.
- Each iteration of the MG solver is an $O(N)$ operation.
- The technique can also be used for the nonlinear and high dimensional equations.

Fifth-order KdV

$$u_t + u_{xxxxx} = 0.$$

We rewrite it as a first order system:

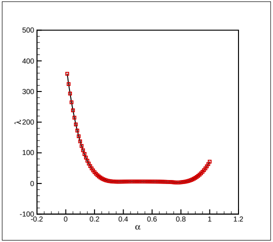
$$u_t + p_x = 0, \quad p - q_x = 0, \quad q - s_x = 0, \quad s - r_x = 0, \quad r - u_x = 0.$$

Applying the LDG method to the system, we obtain an ODE system and apply the time marching method.

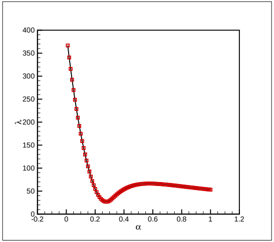
Methods to eliminate the auxiliary variables

- **Method 5X**
- **Method 3X-2X**
- **Method X-2X-2X**

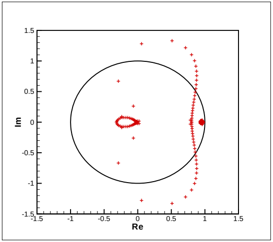
Method 5X: NOT convergent



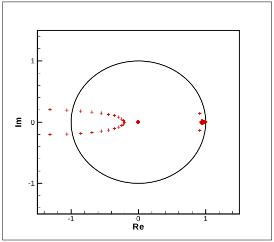
(a) Jacobi, $\mathcal{P}^1$



(b) Gauss-Seidel, $\mathcal{P}^1$

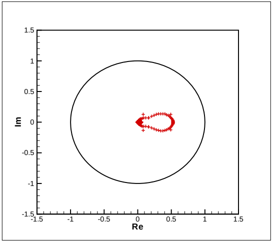


(c) E_h^{2grid}, $\alpha = 0.8$, Jacobi

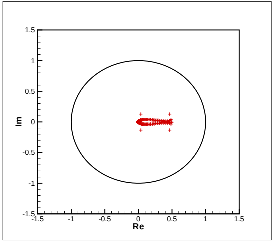


(d) E_h^{2grid}, $\alpha = 0.8$, Gauss-

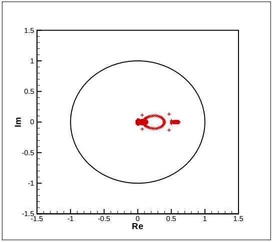
Convergence behavior for the multigrid method



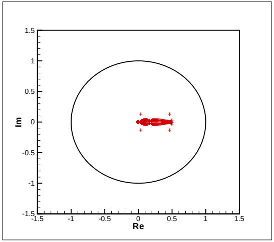
(a) 3X-2X, $\mathcal{P}^1$, E_h^{2grid}



(b) X-2X-2X, $\mathcal{P}^1$, E_h^{2grid}



(c) 3X-2X, $\mathcal{P}^2$, E_h^{2grid}



(d) X-2X-2X, $\mathcal{P}^2$, E_h^{2grid}

Accuracy test for the equation $u_t + u_{xxxxx} = 0$, $\Delta t = 0.1\Delta x$.

		Method 3X-2X				Method X-2X-2X			
	N	L^2 error	order	L^∞ error	order	L^2 error	order	L^∞ error	order
$\mathcal{P}^1$	16	8.85E-01	–	4.99E-01	–	8.86E-01	–	4.99E-01	–
	32	1.76E-01	2.32	1.07E-01	2.21	1.76E-01	2.32	1.07E-01	2.21
	64	2.52E-02	2.80	1.63E-02	2.71	2.52E-02	2.80	1.63E-02	2.71
	128	3.36E-03	2.90	2.38E-03	2.77	3.36E-03	2.90	2.38E-03	2.77
$\mathcal{P}^2$	16	8.02E-01	–	4.54E-01	–	8.02E-01	–	4.54E-01	–
	32	1.55E-01	2.36	8.77E-02	2.37	1.55E-01	2.36	8.77E-02	2.37
	64	2.19E-02	2.82	1.24E-02	2.82	2.19E-02	2.82	1.24E-02	2.82
	128	2.81E-03	2.96	1.59E-03	2.96	2.81E-03	2.96	1.59E-03	2.96

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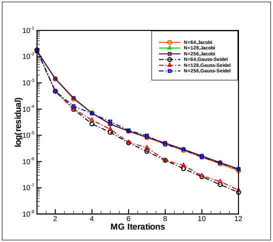
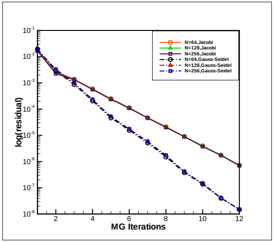
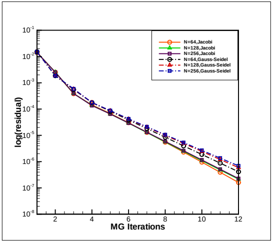
Convergence rates for the MG solver for $\mathcal{P}^1$ and $\mathcal{P}^2$



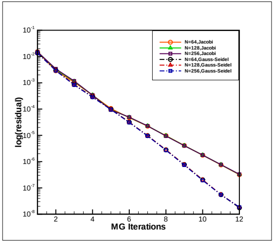
(a) Method 3X-2X, $\mathcal{P}^1$



(b) Method X-2X-2X, $\mathcal{P}^1$



(c) Method 3X-2X, $\mathcal{P}^2$



(d) Method X-2X-2X, $\mathcal{P}^2$

Observation

- High order time discretization methods can be coupled with the LDG space discretization.
- Time step $\Delta t = \mathcal{O}(\Delta x)$.
- The eigenvalue spectra of E_h^{2grid} for Method 5X is larger than 1 and the two-grid algorithm is not convergent.
- The eigenvalue spectra of E_h^{2grid} is strictly less than 1, i. e. the two-grid algorithm is convergent for Method 3X-2X and Method X-2X-2X.
- Method X-2X-2X shows better convergence behavior than Method 3X.
- Method X-2X-2X require more memory, especially for high dimensional case.
- Each iteration of the MG solver is an $O(N)$ operation.
- The technique can also be used for the nonlinear and high dimensional equations.

The general odd-order linear PDEs

$$u_t + u_x^{(2m+1)} = 0.$$

We first rewrite it as a first order system:

$$u_t + (p_{2m})_x = 0, \quad p_{2m} = (p_{2m-1})_x, \ldots, p_2 = (p_1)_x, \quad p_1 = u_x.$$

Applying the LDG method to the system, we obtain an ODE system and apply the time marching method.

Three methods to eliminate the auxiliary variables

- **Method 3X-2X-2X...-2X**
- **Method X-2X-2X...-2X**
  - Better convergence behavior than Method 3X-2X-2X...-2X.
  - Require more memory, especially for high dimensional case.

Conclusion and future work

- Fast solvers for the even and odd equations with LDG discretization
- The numerical convergence behavior of multigrid method is investigated.
- Future work
  - The theoretical analysis for the convergence behavior of multigrid method
  - Proper preconditioner for the high stiff system.

Reference

More information about the algorithm and theoretical analysis can be found in: http://staff.ustc.edu.cn/~yxu/

Thank you!